3D Incompressible Navier-Stokes Equations in a Thin, Spherical Shell

Applying Navier Boundary Conditions to a Thin Domain

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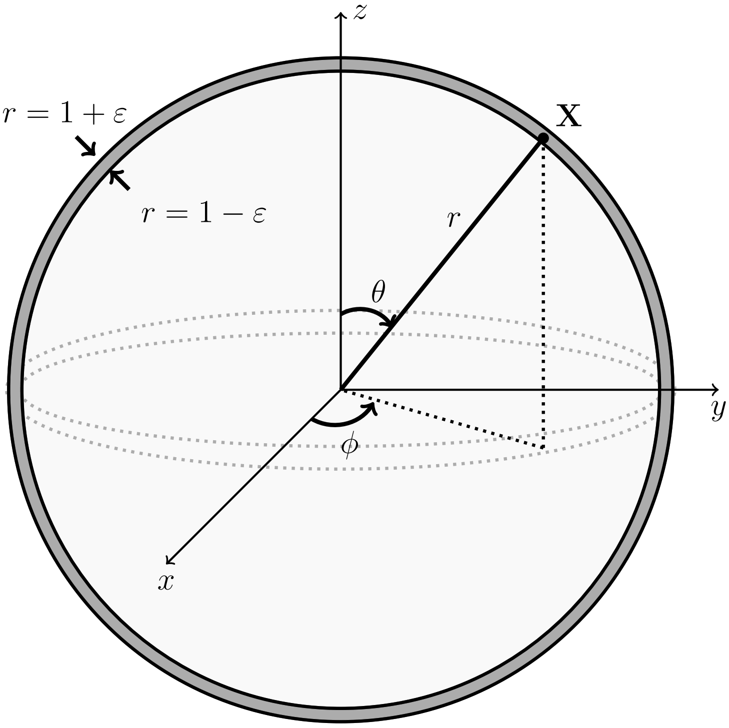

FIGURE 1. Our domain in $\mathbb{R}^3$

$$\Omega := \{\mathbf{x}(r,\theta,\phi) \in \mathbb{R}^3 \,\big|\, r \in [1-\varepsilon, 1+\varepsilon],\, \theta \in [0,\pi],\, \phi \in [0,2\pi)\}$$

Abstract

In this project, we investigate the use of Navier boundary conditions to create an analogue for the atmosphere of the Earth. We are be working on a thin spherical shell domain, where the radius of the inner sphere is much larger than the width of the shell itself (see Figure 1). We will look at different choices of boundary conditions involving the generalised Navier boundary conditions seen in [1]; we begin by exploring the pro's and con's of different forms of the Navier boundary conditions, as well as seeing what results can be obtained from them. There have been numerous works on thin domains, such as [3] by R.Temam & M.Ziane, and some works in regards to Navier boundary conditions in such papers as [2] from D. Iftimie, G. Raugel, and G.Sell. We will tailor the methods of these papers to suit our particular circumstances. We aim to find estimates for the energy $\|\mathbf{u}\|_{L^2(\Omega)}$, and for the enstrophy $\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ where, after finding a restriction on the initial data of the system, we hope to establish an existence theorem for strong solutions of the Navier-Stokes equations.

Contents

0. Introduction

The 3D Navier-Stokes equations (NSE) are prevalent in most areas of fluid dynamics, and are especially key when working with geophysical models. In order to investigate the NSE within the atmosphere, we turn our attention to the idea of thin domains. Thin domains are commonly used throughout mathematics, appearing in solid mechanics, fluid mechanics, physiology, and many other areas. In our geophysical model, since the radius of the Earth is much larger than the height of the atmosphere, we can use thin domains as a good approximation. To this end, we set the edges of our domain to be when $r = 1 + \varepsilon$, and $r = 1 - \varepsilon$, where $\varepsilon > 0$ is a non-dimensional small-scale parameter. As seen in Figure 1, we will work both in Cartesian and spherical polar coordinates, with our domain

$$\begin{aligned}\Omega :=& \{\mathbf{x}(r,\theta,\phi) \in \mathbb{R}^3 |\, r \in (1-\varepsilon, 1+\varepsilon),\, \theta \in [0,\pi],\, \phi \in [0, 2\pi)\} \\ =& \{\mathbf{x}(x,y,z) \in \mathbb{R}^3 |\, 1-\varepsilon < \sqrt{x^2+y^2+z^2} < 1+\varepsilon\}.\end{aligned}$$

We also define the mathematical setting of the NSE as

$$L_\varepsilon := \{\mathbf{v} \in \left(\mathrm{L}^2(\Omega)\right)^3 |\, \operatorname{div} \mathbf{v} = 0 \text{ on } \Omega,\, \mathbf{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega\},$$

and call our boundary $\partial\Omega := \Gamma = \Gamma_b \cup \Gamma_t$, where Γ_b and Γ_t are the inner and outer boundaries, respectively. Throughout the paper, we introduce different choices of boundary condition for our domain; we start off with a generalised Navier boundary condition we have seen in [1],

$$[\boldsymbol{S}\vec{n} + \lambda\mathbf{u}]_{\tan}|_{\Gamma} = \vec{0}, \qquad \text{with } \lambda \geq 0.$$

Then, we set $\lambda = 0$ to get our 'Navier-Navier' boundary conditions

$$[\boldsymbol{S}\vec{n}]_{\tan}|_{\Gamma} = \vec{0}.$$

Finally, we define our 'Non-slip - Navier' boundary conditions

$$(\mathbf{S}\vec{n})_{\tan}|_{\Gamma_t} = \vec{0},$$
$$\mathbf{u}|_{\Gamma_b} = \vec{0}.$$

While there have been papers using this (or an equivalent) domain, such as [3], they utilise other boundary conditions, e.g.

$$\omega \times \vec{n}\big|_{\Gamma} = \vec{0}, \qquad \text{in [3]}.$$

However, our use of the Navier boundary condition is a more physical analogue for our geophysical model. These boundary conditions are used on a thin domain by [2], but their thin domain is flat on the lower boundary, thus not proving a good choice when applied to a geophysical model. These boundary conditions on a thin, spherical domain provide us with estimates for the energy and enstrophy of the system, while also helping to find an existence theorem for strong solutions of our NSE.

Main results. We find that, for both of our 'Navier-Navier' and 'Non-slip - Navier' boundary conditions, we have an estimate on the energy of the flow (1.11)

$$\partial_t \|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = -\mu \|\boldsymbol{S}\|^2_{\mathrm{L}^2(\Omega)}.$$

This allows us to write that $\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(t) \leq \|\mathbf{u}_0\|^2_{\mathrm{L}^2(\Omega)}$, for all $t \geq 0$.
Our 'Non-slip - Navier' boundary conditions allow us to show that we can bound the energy of the flow on the boundary to the enstrophy in the interior (3.1), that is

$$\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} \leq \mathrm{O}(\varepsilon)\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}.$$

We also use our boundary conditions to prove a thin-domain version of Poincaré's inequality for the Stokes pressure (8.5)

$$\|q_s\|^2_{L^2(\Omega)} \le O(\varepsilon^2)\|\nabla q_s\|^2_{L^2(\Omega)}.$$

Similarly, we find (8.4) & (8.6),

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \le O(\varepsilon^2)\|\nabla \mathbf{u}\|^2_{L^2(\Omega)},$$

$$\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \le O(\varepsilon^2)\|\Delta \mathbf{u}\|^2_{L^2(\Omega)}.$$

These results allow us to find an ODE for $\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}$ (5.2),

$$\frac{1}{2}\partial_t\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \le -\mu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{1}{2\mu}\|\mathbf{u}\|^2_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{L^2(\Omega)} + I_\Gamma,$$

where I_Γ is a collection of boundary terms, which we use the 'Non-slip - Navier' boundary conditions to find a bound for in order to analyse the ODE.

A main result in some of the literature is the establishing of an existence theorem. In [3] the restriction is, given a function R_0,

$$\lim_{\varepsilon\to 0}\varepsilon^q R_0(\varepsilon) = 0$$

for some $q > \frac{1}{2}$, and

$$\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)} \le R_0^2(\varepsilon)$$

then strong solutions to the Navier-Stokes equations exist for all time.

Similarly, in [2], the restrictions are, for some k_0, k_1, R^* constants,

$$\|M_h\mathbf{u}_0\|_{L^2(\Omega)} \le k_0,$$

$$\|\nabla\mathbf{u}_0\|_{L^2(\Omega)} \le k_1\varepsilon^{-\frac{1}{2}},$$

then,

$$\|\nabla\mathbf{u}\|_{L^2(\Omega)} \le R^*\varepsilon^{-\frac{1}{2}},$$

where M_h is the horizontal component of the barotropic averaging operator.

In this thesis, we establish our own existence theorem for strong solutions of our NSE,

Theorem 0.1. *There exist positive constants* $c_1, c_2, c_3, \varepsilon_0$ *independent of* ε *such that, for any* $0 < \varepsilon \leq \varepsilon_0$*, with initial data*

$$\mathbf{u}_0 \in \mathrm{V} := \{\mathbf{v}(t,\cdot) \in \left(\mathrm{L}^2(\Omega)\right)^3 |\ \nabla \cdot \mathbf{v} = 0,\ \mathbf{v} \cdot \vec{n}|_\Gamma = 0,\ \mathbf{v}|_{\Gamma_b} = 0,\ [\boldsymbol{S}\vec{n}]_{\tan}|_{\Gamma_t} = \vec{0}\,\},$$

where

$$\|\nabla \mathbf{u}_0\|^2_{\mathrm{L}^2(\Omega)} \leq c_1 \varepsilon^{-\frac{1}{2}}.$$

Then, the incompressible Navier-Stokes equations (1.3) *have solutions* $\mathbf{u}(t,\cdot) \in V$*, such that*

$$\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}(t) \leq c_2 \|\mathbf{u}_0\|_{\mathrm{L}^2(\Omega)},$$
$$\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(t) \leq c_3 \varepsilon^{-\frac{1}{2}}, \qquad \text{for all}\, t \geq 0.$$

Estimates on $\|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}$ and $\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}$ also follow from this theorem.

The rest of the thesis is organised as follows. In Section 1 we set up the problem, finding energy estimates for our NSE and defining a Leray projection on L_ε. Sections 2 & 3 are spent focussing on our choices of boundary condition; Section 2 is an investigation of the 'Navier-Navier' boundary conditions, exploring the identities and problems associated to that particular choice, providing motivation for a change of boundary conditions. Section 3, following Section 2's lead, introduces the 'Non-slip - Navier' boundary conditions, showing how they circumvent the problems established in Section 2, and finding some useful results associated to the boundary conditions. Sections 4 & 5 use the results found so far, along with some well-known inequalities, in order to find estimates for our NSE in the form of an ODE in $\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$, which we then analyse in Section 6. This then provides us with a bound on our enstrophy, provided our initial data satisfies conditions obtained during the analysis period.

The thesis ends with Sections 7 & 8, our Conclusion and Appendix, respectively. In Section 7, we bring together our previous results to form an existence theorem for our NSE. We finish this section with a look at possible future research, presenting a framework based off other literature, as well as suggesting a problem that requires further investigation. We also look at how different literature approach the problem of bounding an H^2 norm by an L^2 norm of the Laplacian operator. Section 8 then brings the project to a close by recalling some classic inequalities, together with proofs not included in the main work.

1. Navier-Stokes equations with Navier boundary conditions

1.1. **Incompressible Navier-Stokes equations.** We can write our spatial domain Ω on a thin spherical shell in Cartesian coordinates as

$$\Omega := \{(x,y,z) \mid \sqrt{x^2+y^2+z^2} \in (1-\varepsilon, 1+\varepsilon)\}. \tag{1.1}$$

with boundary

$$\begin{aligned} \Gamma &:= \Gamma_b \cup \Gamma_t \\ &= \{(x,y,z) \mid \sqrt{x^2+y^2+z^2} = 1 \pm \varepsilon\}. \end{aligned} \tag{1.2}$$

Here, Γ_b and Γ_t are the inner and outer boundaries, respectively.
The incompressible Navier-Stokes equations can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla q = \mu\Delta\mathbf{u}, \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{1.3}$$

subject to the generalised Navier boundary condition for $\lambda \geq 0$

$$\mathbf{u}\cdot\vec{n}\big|_\Gamma = 0, \qquad \text{(zero-flux)} \tag{1.4a}$$

$$[\boldsymbol{S}\vec{n} + \lambda\mathbf{u}]_{\tan}|_\Gamma = \vec{0}, \qquad \text{(no shear stress).} \tag{1.4b}$$

Here, q is the pressure of our system, $\vec{n}$ denotes the outward normal at $\Gamma = \Gamma_b \cup \Gamma_t$, subscript "tan" indicates the tangential component, e.g. $\mathbf{u}_{\tan} = \mathbf{u} - (\mathbf{u}\cdot\vec{n})\vec{n}$, and the stress tensor $\boldsymbol{S}$ is defined as

$$\boldsymbol{S} := \nabla\mathbf{u} + (\nabla\mathbf{u})^\intercal,$$

$$\text{where} \quad \nabla\mathbf{u} := \begin{pmatrix} \partial_x u_1 & \partial_y u_1 & \partial_z u_1 \\ \partial_x u_2 & \partial_y u_2 & \partial_z u_2 \\ \partial_x u_3 & \partial_y u_3 & \partial_z u_3 \end{pmatrix}.$$

Also, note that μ is the dynamic viscosity,

$$\mu \propto \frac{1}{\mathrm{Re}}$$

where Re is the Reynolds number of the flow.

1.2. **Energy estimates.** In order to maintain the dissipation of energy that we would expect from a viscous model, we will see that λ must be restricted within our generalised Navier boundary condition (1.4b).

Proposition 1.1. Consider $\mathbf{u}$ satisfying (1.3) subject to the general Navier boundary condition,

$$[\boldsymbol{S}\vec{n} + \lambda \mathbf{u}]_{\tan}|_{\Gamma} = \vec{0},$$

with $\lambda \geq 0$. Then, the energy $\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}$ is decreasing with time.

Proof of Proposition 1.1. Take the $\mathrm{L}^2(\Omega)$ inner product of $\mathbf{u}$ and (1.3),

$$\begin{aligned} \mu\langle \Delta \mathbf{u}, \mathbf{u}\rangle &= \langle \partial_t \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{u}\rangle + \langle \nabla q, \mathbf{u}\rangle \\ &= \frac{1}{2}\partial_t\|\mathbf{u}\|^2 + \int_\Omega (\frac{1}{2}\mathbf{u}\cdot\nabla|\mathbf{u}|^2 + \mathbf{u}\cdot\nabla q) \qquad (1.5) \\ &= \frac{1}{2}\partial_t\|\mathbf{u}\|^2. \end{aligned}$$

The last line is from $\mathbf{u}\cdot\nabla(\frac{|\mathbf{u}|^2}{2} + q) = \nabla\cdot\left[\mathbf{u}(\frac{|\mathbf{u}|^2}{2} + q)\right]$ for $\operatorname{div}\mathbf{u} = 0$, then using the divergence theorem and applying the zero flux boundary condition $\mathbf{u}\cdot\vec{n}\big|_\Gamma = 0$. Also, for $\operatorname{div}\mathbf{u} = 0$, we can write $\operatorname{div}\boldsymbol{S}$ as

$$\begin{aligned} \operatorname{div}\boldsymbol{S} =& \operatorname{div}\left[\nabla\mathbf{u}\right] + \operatorname{div}\left[(\nabla\mathbf{u})^\intercal\right] \\ =& \Delta\mathbf{u} + \begin{pmatrix} \partial_x(\operatorname{div}\mathbf{u}) \\ \partial_y(\operatorname{div}\mathbf{u}) \\ \partial_z(\operatorname{div}\mathbf{u}) \end{pmatrix} = \Delta\mathbf{u}. \end{aligned}$$

We can also find that,

$$\begin{aligned} \int_\Omega \operatorname{div}\left[\boldsymbol{S}\mathbf{u}\right] &= \int_\Omega \mathbf{u}\cdot\operatorname{div}\boldsymbol{S} + \int_\Omega \boldsymbol{S}\cdot\nabla\mathbf{u}, \\ \Rightarrow \quad \int_\Gamma \vec{n}\cdot(\boldsymbol{S}\cdot\mathbf{u}) &= \langle \operatorname{div}\boldsymbol{S}, \mathbf{u}\rangle + \langle \boldsymbol{S}, \nabla\mathbf{u}\rangle \end{aligned}$$

and so,

$$\langle \Delta\mathbf{u}, \mathbf{u}\rangle = \langle \operatorname{div}\boldsymbol{S}, \mathbf{u}\rangle = -\langle \boldsymbol{S}, \nabla\mathbf{u}\rangle + \int_\Gamma (\boldsymbol{S}\vec{n})\cdot\mathbf{u}. \qquad (1.6)$$

Using our definition $\lambda \geq 0$ and the boundary condition in Proposition (1.1),

$$\int_{\Gamma} (\boldsymbol{S}\vec{n} + \lambda \mathbf{u}) \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \int_{\Gamma} (\boldsymbol{S}\vec{n}) \cdot \mathbf{u} \leq 0, \tag{1.7}$$

since $\mathbf{u}$ is orthogonal to $\vec{n}$ on the boundary by (1.4a). Now, using the definition for inner products of real-valued matrices,

$$\langle \boldsymbol{S}, \nabla \mathbf{u} \rangle = \langle \boldsymbol{S}^{\intercal}, (\nabla \mathbf{u})^{\intercal} \rangle.$$

Then, using the fact that $\boldsymbol{S}$ is symmetric,

$$\langle \boldsymbol{S}, \nabla \mathbf{u} \rangle = \langle \boldsymbol{S}^{\intercal}, (\nabla \mathbf{u})^{\intercal} \rangle = \langle \boldsymbol{S}, (\nabla \mathbf{u})^{\intercal} \rangle,$$

and therefore,

$$\langle \boldsymbol{S}, (\nabla \mathbf{u}) \rangle = \frac{1}{2} \langle \boldsymbol{S}, \nabla \mathbf{u} + (\nabla \mathbf{u})^{\intercal} \rangle = \frac{1}{2} \|\boldsymbol{S}\|^2. \tag{1.8}$$

Thus, by combining (1.6), (1.7), and (1.8), we have

$$\langle \Delta \mathbf{u}, \mathbf{u} \rangle \leq -\frac{1}{2} \|\boldsymbol{S}\|^2. \tag{1.9}$$

This combined with (1.5) gives us,

$$\partial_t \|\mathbf{u}\|^2 \leq -\mu \|\boldsymbol{S}\|^2 \tag{1.10}$$

Hence, the proof is finished. □

In fact, (1.7) is an equality if $\lambda = 0$ or $\mathbf{u}|_{\Gamma} = 0$ and so, for our boundary conditions defined in Sections 2 & 3, we will be working with the equation

$$\partial_t \|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = -\mu \|\boldsymbol{S}\|^2_{\mathrm{L}^2(\Omega)}. \tag{1.11}$$

We would like our energy to be strictly decreasing for $\mathbf{u} \neq 0$ to be consistent with physical dissipation of viscous flows. However, we can see that, for a purely anti-symmetric $\nabla \mathbf{u}$, $\mathbf{S} = 0$ and thus, $\partial_t \|\mathbf{u}\|^2 \leq 0$. In order to prevent this scenario, we would like to implement a condition to exclude this case from our work, something we will discuss more closely in Sections 2 & 3.

1.3. **Leray projections.** Before attempting to find estimates for our Navier-Stokes equations, we would like to have all of the terms in our equation satisfy the basic properties of our flow. That is, for some $\mathbf{v}$

$$\nabla \cdot \mathbf{v} = 0, \quad \text{On } \Omega$$
$$\mathbf{v} \cdot \vec{n}\big|_{\Gamma} = 0, \quad \text{On } \Gamma$$

To achieve this, we will introduce Leray projections for our terms.

Lemma 1.2. *We define $\mathbb{P}$ as a Leray projection, so that, for vector $\mathbf{v} \in \mathbf{C}^\infty$, $\mathbb{P}(\mathbf{v}) = \mathbf{v} + \nabla p$ (for p scalar), where p satifies the conditions*

$$\begin{cases} \Delta p = -\nabla \cdot \mathbf{v}, & \text{On } \Omega \\ \dfrac{\partial p}{\partial \vec{n}} = -\vec{n} \cdot \mathbf{v}, & \text{On } \Gamma \\ \displaystyle\int_\Omega p\, dx = 0, & \text{(No spatial average)} \end{cases}$$

This Leray projection then satisfies

$$\nabla \cdot \mathbb{P} = 0, \qquad \textit{On } \Omega \tag{1.12a}$$

$$\mathbb{P} \cdot \vec{n}\big|_{\Gamma} = 0, \qquad \textit{On } \Gamma \tag{1.12b}$$

Then, the complement of $\mathbb{P}$, $\mathbb{Q} := \mathbb{I} - \mathbb{P}$ where $\mathbb{I}$ is the identity projection, is orthogonal to $\mathbb{P}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ for any $\mathbf{v} \in \mathbf{C}^\infty$. That is to say,

$$\langle \mathbb{P}(\mathbf{v}_1), \mathbb{Q}(\mathbf{v}_2) \rangle_{L^2(\Omega)} = 0 \quad \text{for any } \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{C}^\infty \tag{1.13}$$

Proof of Lemma 1.2. Verification of (1.12a) and (1.12b) is straightforward. To prove (1.13), we see that, for $\mathbb{P}(\mathbf{v}) = \mathbf{v} + \nabla p$, the complement $\mathbb{Q}(\mathbf{v}) := \mathbb{I}(\mathbf{v}) - \mathbb{P}(\mathbf{v}) = \mathbf{v} - \mathbb{P}(\mathbf{v}) = -\nabla p$. Thus, $\mathbb{Q}(\mathbf{v})$ can be written as $-\nabla p$. Hence,

$$\begin{aligned} \langle \mathbb{P}(\mathbf{v}_1), \mathbb{Q}(\mathbf{v}_2) \rangle_{L^2(\Omega)} &= -\int_\Omega (\mathbb{P}(\mathbf{v}_1)) \cdot (\nabla p_2) \\ &= \int_\Omega (\nabla \cdot (\mathbb{P}(\mathbf{v}_1))) p_2 - \int_\Gamma (\vec{n} \cdot (\mathbb{P}(\mathbf{v}_1))) p_2 \\ &= 0 - 0 \end{aligned}$$

The final integrals are found to be 0 from (1.12a) & (1.12b), respectively, assuming the conditions for p define above. Hence, (1.13) is proven. □

1.4. **Our Leray Navier-Stokes equation.** Firstly, we split up the ∇q term into $\nabla q = \nabla q_1 - \mu \nabla q_s$, where q_s is known as the Stokes pressure, transforming (1.3) into

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q_1 = \mu(\Delta \mathbf{u} + \nabla q_s), \quad .$$

which, by defining $\vec{\sigma}_1 := \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q_1$ and $\vec{\sigma}_2 := (\Delta \mathbf{u} + \nabla q_s)$, can be rewritten as

$$\partial_t \mathbf{u} + \vec{\sigma}_1 = \mu \vec{\sigma}_2, \tag{1.14}$$

Assuming $\vec{\sigma}_1, \vec{\sigma}_2$ satisfy (1.4), we find conditions for q_1 and q_s

$$\begin{cases} \Delta q_1 = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}), & \text{On } \Omega \\ \dfrac{\partial q_1}{\partial \vec{n}} = -\vec{n} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}), & \text{On } \Gamma \end{cases} \tag{1.15}$$

$$\begin{cases} \Delta q_s = 0, & \text{On } \Omega \\ \dfrac{\partial q_s}{\partial \vec{n}} = -\Delta \mathbf{u} \cdot \vec{n}, & \text{On } \Gamma \end{cases} \tag{1.16}$$

We can also define $\bar{q}_i$ such that

$$\bar{q}_i = \frac{1}{|\Omega|} \int_\Omega q_i,$$

which is the average value for q_i. Then we can redefine q_i as

$$q_i' = q_i - \bar{q}_i,$$

where q_i' still satisfies the conditions above for q_1', q_s'. Hence, after dropping $'$, we have $\{q_1, q_s\}$ which satisfy the above conditions and have zero spatial average. With these conditions, we now define (σ_1, σ_2) as Leray projections,

$$\sigma_1 = \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}), \qquad \sigma_2 = \mathbb{P}(\Delta \mathbf{u})$$

which satisfy (1.12a) & (1.12b).

This allows us to write down our projected Navier-Stokes equation

$$\partial_t \mathbf{u} + \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}) = \mu \mathbb{P}(\Delta \mathbf{u}), \tag{1.17}$$

where (1.12a) & (1.12b) are automatically satisfied for $t \geq 0$, provided $\nabla \cdot \mathbf{u}_0 = 0$ and $\mathbf{u}_0 \cdot \vec{n}\big|_\Gamma = 0$. Now equipped with our projected Navier-Stokes equation (1.17), we take the inner product with respect to $\Delta \mathbf{u}$.

$$\begin{aligned} \langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle + \langle \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}), \Delta \mathbf{u} \rangle &= \mu \langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u} \rangle, \\ \mathrm{I} \quad + \quad \mathrm{II} \quad &= \quad \mathrm{III} \end{aligned} \tag{1.18}$$

2. 'Navier-Navier' boundary conditions

In order to be able to deal with any boundary terms in our equation, we need to properly define our boundary conditions on Γ. We begin by choosing our 'Navier-Navier' boundary conditions, which is equivalent to setting $\lambda = 0$ in our generalised case.

$$[\boldsymbol{S}\vec{n}]_{\tan}|_\Gamma = \vec{0}.$$

2.1. General Properties.

The 'Navier-Navier' boundary conditions for this system are

$$\begin{aligned} \mathbf{u} \cdot \vec{n}\big|_\Gamma &= 0, \qquad &&\text{(zero-flux)} \\ [\boldsymbol{S}\vec{n}]_{\tan}|_\Gamma &= \vec{0}, &&\text{(no shear stress).} \end{aligned}$$

In order to derive some more physical properties of $\mathbf{u}$ from these conditions, we first remember some important vector identities,

$$\begin{cases} ((\nabla \mathbf{u})\mathbf{u}') \cdot \mathbf{u}'' = ((\mathbf{u}' \cdot \nabla)\mathbf{u})\mathbf{u}'' \\ ((\nabla \mathbf{u})^\intercal \mathbf{u}') \cdot \mathbf{u}'' = (\mathbf{u}'' \cdot \nabla \mathbf{u}) \cdot \mathbf{u}'. \end{cases}$$

Taking $\mathbf{u}' = \vec{n}$, $\mathbf{u}'' = \vec{\tau}$, where $\vec{\tau}$ is a tangent vector on Γ, we then find

$$\begin{cases} ((\nabla \mathbf{u})\vec{n}) \cdot \vec{\tau} = ((\vec{n} \cdot \nabla)\mathbf{u})\vec{\tau} \\ ((\nabla \mathbf{u})^\intercal \vec{n}) \cdot \vec{\tau} = (\vec{\tau} \cdot \nabla \mathbf{u}) \cdot \vec{n}, \end{cases}$$

which, if we add both equations, provides us with

$$\begin{aligned} [\boldsymbol{S}\vec{n}] \cdot \vec{\tau} &= ((\nabla \mathbf{u})\vec{n}) \cdot \vec{\tau} + ((\nabla \mathbf{u})^\intercal \vec{n}) \cdot \vec{\tau} \\ &= ((\vec{n} \cdot \nabla)\mathbf{u})\vec{\tau} + (\vec{\tau} \cdot \nabla \mathbf{u}) \cdot \vec{n} \end{aligned}$$

It can be shown that $(\vec{\tau}\cdot\nabla\mathbf{u})\cdot\vec{n} = -(\vec{\tau}\cdot\nabla\vec{n})\cdot\mathbf{u}$ and, by noting that $(\operatorname{curl}\mathbf{u})\times\vec{n} = ((\nabla\mathbf{u}) - (\nabla\mathbf{u})^{\intercal})\cdot\vec{n}$, one can find

$$\begin{aligned}
[\boldsymbol{S}\vec{n}]\cdot\vec{\tau} &= ((\vec{n}\cdot\nabla)\mathbf{u})\vec{\tau} + (\vec{\tau}\cdot\nabla\mathbf{u})\cdot\vec{n} \\
&= ((\vec{n}\cdot\nabla)\mathbf{u})\vec{\tau} - (\vec{\tau}\cdot\nabla\vec{n})\cdot\mathbf{u} \\
&= ((\operatorname{curl}\mathbf{u})\times\vec{n})\cdot\vec{\tau} - 2(\vec{\tau}\cdot\nabla\vec{n})\cdot\mathbf{u}.
\end{aligned}$$

We can define the **second fundamental form** $\mathscr{X}$ of two tangent vectors $\vec{\tau}_1, \vec{\tau}_2$ as

$$\mathscr{X}(\vec{\tau}_1, \vec{\tau}_2) := -(\vec{\tau}_1\cdot\nabla\vec{n})\vec{\tau}_2,$$

which can be shown to be symmetric in $\vec{\tau}_1, \vec{\tau}_2$. Therefore, since $\mathbf{u}$ is a tangent vector on Γ from (1.4a), we find

$$\begin{aligned}
-(\vec{\tau}\cdot\nabla\vec{n})\mathbf{u}\big|_\Gamma &= -(\mathbf{u}\cdot\nabla\vec{n})\vec{\tau}, \\
\Rightarrow \quad [\boldsymbol{S}\vec{n}]\cdot\vec{\tau}|_\Gamma &= ((\vec{n}\cdot\nabla)\mathbf{u})\vec{\tau} - (\mathbf{u}\cdot\nabla\vec{n})\cdot\vec{\tau} \\
&= ((\operatorname{curl}\mathbf{u})\times\vec{n})\cdot\vec{\tau} - 2(\mathbf{u}\cdot\nabla\vec{n})\cdot\vec{\tau}.
\end{aligned}$$

Hence, changing $\vec{\tau}$ to a generalised tangential direction,

$$[\boldsymbol{S}\vec{n}]_{\tan}|_\Gamma = ((\vec{n}\cdot\nabla)\mathbf{u})_{\tan} - (\mathbf{u}\cdot\nabla\vec{n}) \tag{2.1a}$$

$$= ((\operatorname{curl}\mathbf{u})\times\vec{n}) - 2(\mathbf{u}\cdot\nabla\vec{n}). \tag{2.1b}$$

2.2. **Geometry of 'Navier-Navier' boundary conditions.** Two common choices of boundary conditions from the literature are

$$\omega\times\vec{n}\big|_\Gamma = 0 \qquad \text{and} \qquad (\vec{n}\cdot\nabla\mathbf{u})_{\tan}\big|_\Gamma = 0.$$

However, from (2.1), we found that,

$$\begin{aligned}
[\boldsymbol{S}\vec{n}]_{\tan}|_\Gamma &= (\vec{n}\cdot\nabla\mathbf{u})_{\tan} - \mathbf{u}\cdot\nabla\vec{n} \\
&= \omega\times\vec{n} - 2(\mathbf{u}\cdot\nabla\vec{n}),
\end{aligned}$$

so, for either of these boundary conditions, we end up with

$$[\boldsymbol{S}\vec{n} + C(\mathbf{u}\cdot\nabla\vec{n})]_{\tan}|_\Gamma = 0,$$

where $C \in \{1,2\}$ (C depends on which boundary condition is chosen). Hence, $C(\mathbf{u}\cdot\nabla\vec{n}) = \lambda\mathbf{u}$.

However, from [1] we see that, for a surface with basis coordinates $(\mathbf{e}_1, \mathbf{e}_2, \vec{n})$,

$$\mathbf{u} = \mathbf{e}_1(\mathbf{u}\cdot\mathbf{e}_1) + \mathbf{e}_2(\mathbf{u}\cdot\mathbf{e}_2)$$
$$-\mathbf{u}\cdot\nabla\vec{n} = \kappa_1\mathbf{e}_1(\mathbf{u}\cdot\mathbf{e}_1) + \kappa_2\mathbf{e}_2(\mathbf{u}\cdot\mathbf{e}_2).$$

Hence, on the boundary $r = 1 \pm \varepsilon$,

$$\kappa_1 = \kappa_2 = \mp\frac{1}{r},$$

Also, by (2.1a),

$$(\vec{n}\cdot\nabla\mathbf{u})_{\tan} = \mathbf{u}\cdot\nabla\vec{n} = \pm\frac{1}{r}\mathbf{u}, \quad \text{where } r = 1\pm\varepsilon. \tag{2.2}$$

Therefore, for the boundary conditions mentioned from the literature, $\lambda = \pm\frac{C}{r}$. However, in either case, $\lambda < 0$ for some part of Γ, meaning that the energy of the flow isn't always decreasing. Hence, we believe our boundary conditions in (1.4b) are more physical on our spherical shell domain Ω, as $\lambda = 0$ ensures that the energy of the system is always decreasing.

2.3. **Problems with our boundary conditions.** When obtaining our energy estimate in Proposition 1.1, we are provided with the inequality

$$\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\mu\|\boldsymbol{S}\|^2_{\mathrm{L}^2(\Omega)}$$

where $\boldsymbol{S} := \nabla\mathbf{u} + (\nabla\mathbf{u})^\intercal$. This is for the case when $\lambda \geq 0$. Using our 'Navier-Navier' boundary conditions, one can find that,

$$\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = -\mu\|\boldsymbol{S}\|^2_{\mathrm{L}^2(\Omega)}$$

For the dissipation of the energy of our flow, we would like

$$\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} < 0 \qquad \Rightarrow \|\boldsymbol{S}\|^2_{\mathrm{L}^2(\Omega)} > 0.$$

However, one can see that there exists $\mathbf{u}$ within the kernel of our inequality,

$$\mathbf{u} = \vec{A} + \vec{B}\times\mathbf{x} \iff (\nabla\mathbf{u})^\intercal = -\nabla\mathbf{u} \iff \boldsymbol{S} = 0,$$

where $\vec{A}, \vec{B}$ are constant 3x1 vectors, and $\mathbf{x} = (x, y, z)^{\intercal}$. It can be shown that, in order to satisfy the 'Navier-Navier' boundary conditions on our domain, we must have $\vec{A} = \vec{0}$. Thus, in order to have a strictly decreasing energy, we need to introduce an extra condition on our flow.

In the literature, we see that this problem can be solved in various ways:

- In the paper [2], the motivation for this problem is the use of the Uniform Korn Inequality

$$c_0 \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} \leq \|\mathbf{S}\|^2_{\mathrm{L}^2(\Omega)} \leq c_0^* \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)},$$

  which clearly breaks down for the case of $\mathbf{u} = \vec{A} + \vec{B} \times \mathbf{x}$. In order to solve this issue, condition (H) is imposed on $\mathbf{u}$, exempting this special case (see Section 2 of [2] for more details).
- In Temam & Ziane's work [3], the problem is circumvented by their choice of boundary conditions. By defining $\operatorname{curl} \mathbf{u} \times \vec{n}|_\Gamma = \vec{0}$, and $\mathbf{u} \cdot \vec{n}|_\Gamma = 0$, the special case of $\mathbf{u} = \vec{A} + \vec{B} \times \mathbf{x}$ is reduced to $\vec{A} = \vec{B} = \vec{0}$.
- For our case, this problem will motivate us to change our boundary conditions in order to exempt the special case of $\mathbf{u} = \vec{B} \times \mathbf{x}$.

Note: this special case for $\mathbf{u}$ also means $\Delta \mathbf{u} = 0$, so it becomes trivial to take the inner product $\langle \Delta \mathbf{u}, \partial_t \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)}$ as seen in certain types of energy estimates for $\|\mathbf{u}\|_{\mathrm{H}^1(\Omega)}$ (see Section 5 of [2]).

3. 'Non-slip - Navier' boundary conditions

By refining our flow to have a non-slip boundary condition on Γ_b, we will try to fix the problems explored above, as well as deriving some results that will be used later.

3.1. **General Properties.** In order to work with our boundary terms in our ODE, we decide to replace the Navier boundary condition with the non-slip boundary condition on our lower boundary,

$$[\boldsymbol{S}\vec{n}]_{\tan}|_{\Gamma_t} = \vec{0},$$

$$\mathbf{u}\big|_{\Gamma_b} = \vec{0},$$

where $\Gamma_b = \big\{(r,\theta,\phi) \in \Omega | \quad r = 1-\varepsilon\big\}$.
Then,

$$\mathbf{u}\cdot\nabla\vec{n}\big|_{\Gamma_t} = \frac{1}{r}\mathbf{u}\big|_{\Gamma_t}$$

With these new boundary conditions, it is worth verifying that our new choice of boundary conditions solves our dissipation problem. Our problem, as we saw in Subsection 2.3, is the existence of $\mathbf{u} \neq 0$, such that

$$\mathbf{u} = \vec{B}\times\mathbf{x} \iff (\nabla\mathbf{u})^{\intercal} = -\nabla\mathbf{u} \iff \boldsymbol{S} = 0,$$

However, for $\mathbf{u}|_{\Gamma_b} = 0$, we would have

$$\mathbf{u} = \begin{pmatrix} b_2 z - b_3 y \\ b_3 x - b_1 z \\ b_1 y - b_2 x \end{pmatrix} = 0 \qquad \text{on}\,\Gamma_b$$

Since this must be true for constant $\vec{B}$, we can pick points on Γ_b in order to solve for b_i.

$$\mathbf{x} = \begin{pmatrix} 1-\varepsilon \\ 0 \\ 0 \end{pmatrix} \implies b_3 = 0,\, b_2 = 0$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1-\varepsilon \\ 0 \end{pmatrix} \implies b_3 = 0,\, b_1 = 0$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1-\varepsilon \end{pmatrix} \implies b_2 = 0,\, b_1 = 0$$

Thus, we find that $b_i = 0$. Hence, the only way $\mathbf{u} = \vec{B}\times\mathbf{x}$ is if $\vec{B} = 0$.
Therefore, for our 'Non-slip - Navier' boundary conditions,

$$\boldsymbol{S} = 0 \iff \mathbf{u} = 0,$$

and, thus our dissipation problem is solved.

3.2. **Useful results.** Using our new boundary conditions, we want to try and find a bound for the H^1 norm of $\mathbf{u}$, so we return to the inner product

$$\langle \mathbb{P}(\Delta \mathbf{u}), \mathbf{u}\rangle_{L^2(\Omega)} = -\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \frac{1}{r}\|\mathbf{u}\|^2_{L^2(\Gamma_t)}.$$

Taking a look at the upper boundary term, we rewrite $\|\mathbf{u}\|_{L^2(\Gamma_t)}$ as

$$|\mathbf{u}|_{(\Gamma_t)} = \Big|_{r=1-\varepsilon}^{r=1+\varepsilon} |\mathbf{u}| = \int_{1-\varepsilon}^{1+\varepsilon} (\frac{\partial}{\partial r}\mathbf{u})\mathrm{d}r$$

then,

$$|\mathbf{u}|^2_{(\Gamma_t)} = \Big[\int_{1-\varepsilon}^{1+\varepsilon} (\frac{\partial}{\partial r}\mathbf{u})\mathrm{d}r\Big]^2 \leq \int_{1-\varepsilon}^{1+\varepsilon} 1^2\mathrm{d}r \cdot \int_{1-\varepsilon}^{1+\varepsilon} (\frac{\partial}{\partial r}\mathbf{u})^2\mathrm{d}r \qquad \text{(by Cauchy-Schwarz (8.1))}.$$

Now, looking at the definition of the L^2 norm,

$$\begin{aligned}
\|\mathbf{u}\|^2_{L^2(\Gamma_t)} &= \int\int_{\Gamma_t} (1+\varepsilon)^2 \sin\theta |\mathbf{u}|^2 \mathrm{d}\theta\mathrm{d}\phi \\
&\leq \int\int_{\Gamma_t} (1+\varepsilon)^2 \sin\theta \Big[\mathrm{O}(\varepsilon)\int_{1-\varepsilon}^{1+\varepsilon} (\frac{\partial}{\partial r}\mathbf{u})^2\mathrm{d}r\Big]\mathrm{d}\theta\mathrm{d}\phi \\
&\leq \int\int\int_{\Omega} r^2\sin\theta\Big[\mathrm{O}(\varepsilon)\frac{(1+\varepsilon)^2}{r^2}(\frac{\partial}{\partial r}\mathbf{u})^2\Big]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&\leq \mathrm{O}(\varepsilon)\Big\|\frac{\partial \mathbf{u}}{\partial r}\Big\|^2_{L^2(\Omega)} \leq \mathrm{O}(\varepsilon)\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \qquad \text{for some } |\varepsilon| << 1
\end{aligned}$$

Hence,

$$\|\mathbf{u}\|^2_{L^2(\Gamma_t)} \leq \mathrm{O}(\varepsilon)\|\nabla \mathbf{u}\|^2_{L^2(\Omega)}, \tag{3.1}$$

and so,

$$\begin{aligned}
\langle \mathbb{P}(\Delta \mathbf{u}), \mathbf{u}\rangle_{L^2(\Omega)} &= -\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \frac{1}{r}\|\mathbf{u}\|^2_{L^2(\Gamma_t)} \\
&\leq -\alpha\|\nabla \mathbf{u}\|^2_{L^2(\Omega)}.
\end{aligned}$$

for $\alpha := 1 - \mathrm{O}(\varepsilon)$.

We can now use our earlier result (1.5) to see

$$\frac{1}{2}\partial_t\|\mathbf{u}\|^2_{L^2(\Omega)} = \mu\langle \mathbb{P}(\Delta \mathbf{u}), \mathbf{u}\rangle_{L^2(\Omega)} \leq -\mu\alpha\|\nabla \mathbf{u}\|^2_{L^2(\Omega)}$$

$$\Rightarrow \qquad \frac{1}{2}\partial_t\|\mathbf{u}\|^2_{L^2(\Omega)} + \mu\alpha\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \leq 0.$$

Integrating over time $t \in [0,T]$, we find

$$\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)}(T) - \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)}(0) + \mu\alpha\int_0^T \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \leq 0$$

We can rewrite this as

$$\mu\alpha \int_0^{\mathrm{T}} \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \le -\frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(\mathrm{T}) + \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(0)$$
$$\le \frac{1}{2}\|\mathbf{u}_0\|^2_{\mathrm{L}^2(\Omega)}\Big|_0^{\mathrm{T}}.$$

Thus,

$$\int_0^{\mathrm{T}} \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \le \frac{1}{2\mu\alpha}\|\mathbf{u}_0\|^2_{\mathrm{L}^2(\Omega)}\Big|_0^{\mathrm{T}} \tag{3.2}$$

Another useful result that we can find using our new boundary conditions is a thin domain analogue for the Poincaré inequality. In order to do this, we will follow Lemma 2.1 in [3] quite closely and adapt it to cover our particular situation.

Proposition 3.1. Let $\mathbf{u} \in \{\mathbf{u} \in \mathrm{L}^2(\Omega) \cap \mathrm{H}^1(\Omega) |\, \mathrm{div}(\mathbf{u}) = 0\,,\, \mathbf{u}\cdot\vec{n}\big|_{\Gamma} = 0,\, \mathbf{u}|_{\Gamma_b} = \mathbf{0}\}$ and $0 \le \varepsilon \le \frac{1}{2}$.
Then,

$$\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \le 4\varepsilon^2 \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$$

Proof. By density, we will prove this for a smooth function $\boldsymbol{\psi} \in \mathrm{C}^\infty(\Omega)$.
For any ξ, η in $[1-\varepsilon, 1+\varepsilon]$:

$$\xi^2\boldsymbol{\psi}^2(x',\xi) + \eta^2\boldsymbol{\psi}^2(x',\eta) = 2\xi\eta\boldsymbol{\psi}(x',\xi)\boldsymbol{\psi}(x',\eta) + \big(\xi\boldsymbol{\psi}(x',\xi) - \eta\boldsymbol{\psi}(x',\eta)\big)^2$$
$$= 2\xi\eta\boldsymbol{\psi}(x',\xi)\boldsymbol{\psi}(x',\eta) + \Big(\int_\eta^\xi \frac{\partial(r\boldsymbol{\psi})}{\partial r}(x',r)\mathrm{d}r\Big)^2,$$

where $x' = x/|x| \in \mathrm{S}^2$. By fixing ξ and integrating with respect to η over $[1-\varepsilon, 1+\varepsilon]$, we find

$$2\varepsilon\xi^2\boldsymbol{\psi}^2(x',\xi) + \int_{1-\varepsilon}^{1+\varepsilon} \eta^2\boldsymbol{\psi}^2(x',\eta)\mathrm{d}\eta = 2\xi\boldsymbol{\psi}(x',\xi)\int_{1-\varepsilon}^{1+\varepsilon} \eta\boldsymbol{\psi}(x',\eta)\mathrm{d}\eta$$
$$+ \int_{1-\varepsilon}^{1+\varepsilon} \Big(\int_\eta^\xi \frac{\partial(r\boldsymbol{\psi})}{\partial r}(x',r)\mathrm{d}r\Big)^2\mathrm{d}\eta.$$

Setting $\xi = 1-\varepsilon$, we then apply this when $\boldsymbol{\psi} = u_r$ - the radial component of our flow - where we have $u_r(x', 1-\varepsilon) = 0$. Then,

$$\int_{1-\varepsilon}^{1+\varepsilon} \eta^2\boldsymbol{\psi}^2(x',\eta)\mathrm{d}\eta = \int_{1-\varepsilon}^{1+\varepsilon} \Big(\int_{1-\varepsilon}^{\eta} \frac{\partial(r\boldsymbol{\psi})}{\partial r}(x',r)\mathrm{d}r\Big)^2\mathrm{d}\eta.$$

We can see that, by setting $\boldsymbol{\psi} = u_\theta$ or $\boldsymbol{\psi} = u_\phi$, we also obtain the same equality since $u_\theta(x', 1-\varepsilon) = u_\phi(x', 1-\varepsilon) = 0$. Then, using the Cauchy-Schwarz (8.1) inequality,

$$\begin{aligned}
\int_{1-\varepsilon}^{1+\varepsilon} \eta^2 \boldsymbol{\psi}^2(x', \eta) \mathrm{d}\eta &= \int_{1-\varepsilon}^{1+\varepsilon} \Big(\int_{1-\varepsilon}^{\eta} \frac{\partial (r\boldsymbol{\psi})}{\partial r}(x', r) \mathrm{d}r \Big)^2 \mathrm{d}\eta \\
&\leq \int_{1-\varepsilon}^{1+\varepsilon} |\xi - \eta| \, \mathrm{d}\eta \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (r\boldsymbol{\psi})}{\partial r}|^2 \mathrm{d}r \\
&\leq 2\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (r\boldsymbol{\psi})}{\partial r}|^2 \mathrm{d}r \\
&\leq 2\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (\boldsymbol{\psi})}{\partial r}|^2 \, r^2 \, \mathrm{d}r + 2\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |\boldsymbol{\psi}|^2 \mathrm{d}r \\
&\leq 2\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (\boldsymbol{\psi})}{\partial r}|^2 \, r^2 \, \mathrm{d}r + 4\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |r\boldsymbol{\psi}|^2 \mathrm{d}r
\end{aligned}$$

Then, relabelling η as r we have, for $\epsilon \leq \frac{1}{2}$

$$\begin{aligned}
\int_{1-\varepsilon}^{1+\varepsilon} r^2 \boldsymbol{\psi}^2(x', \eta) \mathrm{d}r &\leq \frac{2\varepsilon^2}{1 - 4\varepsilon^2} \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (\boldsymbol{\psi})}{\partial r}|^2 \, r^2 \, \mathrm{d}r \\
&\leq 4\varepsilon^2 \int_{1-\varepsilon}^{1+\varepsilon} |\frac{\partial (\boldsymbol{\psi})}{\partial r}|^2 \, r^2 \, \mathrm{d}r
\end{aligned}$$

Integrating with respect to θ and ϕ and using the fact that

$$\int_\Omega |\frac{\partial (\mathbf{u})}{\partial r}|^2 \, \mathrm{d}x \leq \int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}x$$

we have

$$\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq 4\varepsilon^2 \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$$

□

We will use these results in obtaining estimates for our Navier-Stokes equations.

4. Estimates for Leray Navier-Stokes terms

We recall the inner product of the Leray Navier-Stokes equations with $\Delta \mathbf{u}$ seen in (1.18),

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle + \langle \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}), \Delta \mathbf{u} \rangle = \mu \langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u} \rangle,$$

$$\mathrm{I} \quad + \quad \mathrm{II} \quad = \quad \mathrm{III}.$$

We will now find estimates on each of these terms.

4.1. **(I) term.** We begin by first estimating the time derivative term

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)}.$$

We will tackle this term with 3 different scenarios: using no boundary conditions, using our 'Navier-Navier' boundary conditions, and using our 'Non-slip - Navier' boundary conditions.

Without boundary conditions. We use integration by parts to find

$$\begin{aligned}\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} &= -\langle \partial_t \nabla \mathbf{u}, \nabla \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) \\ &= -\frac{1}{2} \partial_t \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}),\end{aligned}$$

Since we have no way of dealing with the above integral in our current scenario, we leave the boundary term for the time being.

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2} \partial_t \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) \tag{4.1}$$

A note on other choices of boundary conditions. Before applying our 'Navier-Navier' boundary conditions to the problem, we are given a good illustration of the positives to selecting other boundary conditions over our own.

As we have just seen, we have

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2} \partial_t \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u})$$

In some literature, a convenient choice of boundary condition is to set

$$\vec{n} \cdot \nabla \mathbf{u}\big|_\Gamma = \frac{\partial \mathbf{u}}{\partial \vec{n}}\Big|_\Gamma = 0,$$

resulting in the simplified case of

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2} \partial_t \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}.$$

In some other literature, if we note that (for an incompressible flow) $\Delta \mathbf{u} = -\nabla \times (\nabla \times \mathbf{u})$, we can write,

$$\begin{aligned}\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} &= -\int_\Omega \partial_t (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) + \int_\Gamma \vec{n} \cdot ((\nabla \times \mathbf{u}) \times \partial_t \mathbf{u}) \\ &= -\partial_t \|\nabla \times \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n} \times (\nabla \times \mathbf{u})) \cdot (\partial_t \mathbf{u}).\end{aligned}$$

This highlights another common choice for a boundary condition, namely,

$$\vec{n} \times (\nabla \times \mathbf{u})\big|_{\Gamma} = 0$$

.

$$\Rightarrow \qquad \langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\partial_t \| \nabla \times \mathbf{u} \|^2_{\mathrm{L}^2(\Omega)}$$

Both of these choices of boundary condition are helpful at simplifying $\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)}$ and $\langle \partial_t \nabla \mathbf{u}, \nabla \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)}$ into non-negative terms (that we can find upper bounds for, at a later stage).

However, as we saw in Subsection 2.2, we feel that our choice of (1.4b) as a boundary condition is more physical on a spherical shell. (Both our 'Navier-Navier' and 'Non-slip-Navier' boundary conditions are particular cases of (1.4b) and, as such, keep $\lambda \geq 0$ for Proposition 1.1)

With 'Navier-Navier' boundary conditions. As we saw earlier, we have

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2} \partial_t \| \nabla \mathbf{u} \|^2_{\mathrm{L}^2(\Omega)} + \int_{\Gamma} (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}).$$

In the case of our 'Navier-Navier' boundary conditions, we see that

$$(\vec{n} \cdot \nabla \mathbf{u})_{\tan} = \mathbf{u} \cdot \nabla \vec{n},$$

and, from (2.2)

$$\mathbf{u} \cdot \nabla \vec{n} = \pm \frac{1}{r} \mathbf{u}.$$

Now, since $\partial_t \mathbf{u}$ is tangential to the boundary,

$$\begin{aligned} \int_{\Gamma} (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) &= \int_{\Gamma} (\mathbf{u} \cdot \nabla \vec{n}) \cdot (\partial_t \mathbf{u}) \\ &= \int_{\Gamma} (\pm \frac{1}{r} \mathbf{u}) \cdot (\partial_t \mathbf{u}) \\ &= \pm \frac{1}{r} \langle \mathbf{u}, \partial_t \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = \pm \frac{1}{2r} \partial_t \| \mathbf{u} \|^2_{\mathrm{L}^2(\Gamma)}. \end{aligned}$$

Hence, with our 'Navier-Navier' boundary conditions on $\Gamma = \Gamma_b \cup \Gamma_t$,

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2} \partial_t \| \nabla \mathbf{u} \|^2_{\mathrm{L}^2(\Omega)} \pm \frac{1}{2r} \partial_t \| \mathbf{u} \|^2_{\mathrm{L}^2(\Gamma)}. \tag{4.2}$$

With ‘Non-slip - Navier’ boundary conditions. With our ‘Non-slip - Navier’ boundary conditions, we see that

$$\mathbf{u} \cdot \nabla \vec{n} = \frac{1}{r}\mathbf{u}$$

on Γ_t, and so

$$\begin{aligned}\int_{\Gamma} (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) &= \int_{\Gamma_t} (\vec{n} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) \\ &= \frac{1}{2r} \partial_t \|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)}.\end{aligned}$$

Thus, using our ‘Non-slip - Navier’ boundary conditions,

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2}\partial_t \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2r}\partial_t \|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)}. \tag{4.3}$$

4.2. **(II) term.** For our second term

$$\langle \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}), \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)}$$

we don’t have to worry about boundary terms like with the other parts of (1.18). Rather, we will need to use some inequalities that are only defined for particular boundary conditions and, as such, we will divide our working in a similar fashion to Subsection 4.1.

Without boundary conditions. For the term in (1.18) as seen above, we can ”transfer” our projection across our inner product,

$$\langle \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}), \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = \langle \mathbf{u} \cdot \nabla \mathbf{u},\, \mathbb{P}(\Delta \mathbf{u}) \rangle_{\mathrm{L}^2(\Omega)}$$

Then, we can find the inequality

$$\langle \mathbf{u} \cdot \nabla \mathbf{u}, \mathbb{P}(\Delta \mathbf{u}) \rangle_{\mathrm{L}^2(\Omega)} \leq \|\mathbf{u}\|_{\mathrm{L}^p(\Omega)} \|\nabla \mathbf{u}\|_{\mathrm{L}^q(\Omega)} \|\mathbb{P}(\Delta \mathbf{u})\|_{\mathrm{L}^r(\Omega)} \quad \text{where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

using a combination of Hölder’s (8.2) and the Cauchy-Schwarz (8.1) inequality. Taking $p \to \infty$ and $r = q = 2$, we have

$$\begin{aligned}\langle \mathbf{u} \cdot \nabla \mathbf{u}, \mathbb{P}(\Delta \mathbf{u}) \rangle_{\mathrm{L}^2(\Omega)} &\leq \|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \|\nabla \mathbf{u}\|_{\mathrm{L}^2(\Omega)} \|\mathbb{P}(\Delta \mathbf{u})\|_{\mathrm{L}^2(\Omega)} \\ &\leq \|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \|\nabla \mathbf{u}\|_{\mathrm{L}^2(\Omega)} \|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

Finally, applying Young’s inequality (8.3),

$$\langle \mathbb{P}\mathbf{u} \cdot \nabla \mathbf{u}, (\Delta \mathbf{u}) \rangle_{\mathrm{L}^2(\Omega)} \leq \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)} \|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}.$$

In order to deal with our L^∞ norm, we need to apply some of our boundary conditions to the problem. Hence, without any boundary conditions, we have

$$\langle \mathbb{P}\mathbf{u}\cdot\nabla\mathbf{u},(\Delta\mathbf{u})\rangle_{\mathrm{L}^2(\Omega)} \leq \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}. \tag{4.4}$$

With 'Navier-Navier' boundary conditions. From the (4.4), we have

$$\langle \mathbb{P}\mathbf{u}\cdot\nabla\mathbf{u},(\Delta\mathbf{u})\rangle_{\mathrm{L}^2(\Omega)} \leq \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}.$$

In order to work with our L^∞ norm, we need to use Agmon's inequality (8.7), and our Poincaré inequality (8.4), both of which require a zero point on our flow. Hence, we leave the inequality as it is and move on to our final boundary conditions.

With 'Non-slip - Navier' boundary conditions. We saw, in the derivation of (4.4), that

$$\langle \mathbf{u}\cdot\nabla\mathbf{u},\mathbb{P}(\Delta\mathbf{u})\rangle_{\mathrm{L}^2(\Omega)} \leq \|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|_{\mathrm{L}^2(\Omega)}.$$

Then, using Agmon's inequality, (8.7), we can find that

$$\|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \leq \tilde{C}\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|^{\frac{3}{4}}_{\mathrm{L}^2(\Omega)}$$

$$\Rightarrow \qquad \langle \mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq \tilde{C}\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\|\nabla\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|^{\frac{7}{4}}_{\mathrm{L}^2(\Omega)}$$

Using the Poincaré (8.4) and Young's (8.3) inequalities , we can write

$$\begin{aligned}\tilde{C}\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\|\nabla\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|^{\frac{7}{4}}_{\mathrm{L}^2(\Omega)} &\leq \mathrm{O}(\varepsilon^{\frac{1}{4}})\|\nabla\mathbf{u}\|^{\frac{5}{4}}_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|^{\frac{7}{4}}_{\mathrm{L}^2(\Omega)}\\ &\leq \frac{\mathrm{O}(\varepsilon^{\frac{1}{4}})}{s}\Big[\frac{1}{\eta}\|\nabla\mathbf{u}\|^{\frac{5}{4}}_{\mathrm{L}^2(\Omega)}\Big]^s + \frac{\mathrm{O}(\varepsilon^{\frac{1}{4}})}{r}\Big[\eta\|\Delta\mathbf{u}\|^{\frac{7}{4}}_{\mathrm{L}^2(\Omega)}\Big]^r,\end{aligned}$$

$$\text{where} \quad \frac{1}{s}+\frac{1}{r}=1.$$

Taking $r=\frac{8}{7}, s=8$, we obtain

$$\langle \mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq \frac{\mathrm{O}(\varepsilon^{\frac{1}{4}})}{8\eta^8}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)} + \frac{7\eta^{\frac{8}{7}}\mathrm{O}(\varepsilon^{\frac{1}{4}})}{8}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)},$$

which, when $\eta = \Big(\dfrac{4\mu}{7\,\mathrm{O}(\varepsilon^{\frac{1}{4}})}\Big)^{\frac{7}{8}}$, can be written as

$$\mathrm{O}\Big(\frac{\varepsilon^2}{\mu^7}\Big)\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$$

Thus,

$$\langle \mathbb{P}(\mathbf{u}\cdot\nabla\mathbf{u}),\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \tag{4.5}$$

A quick note on Agmon's inequality. The version of Agmon's inequality we are using is ascertained in [3] independently of boundary conditions. However, we feel that the assumption $\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)} \leq C\|\Delta\mathbf{u}\|_{\mathrm{L}^2(\Omega)}$, with constant C independent of ε is not something we can just assume for our situation. This is proven in [2] & [6] for domains and/or boundary conditions different from our choices here. We also have, from [6], the classical Cattabriga-Solonnikov "H^2 - regularity" inequality

$$\sum_{i,j=1}^{3} |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2_{\mathrm{L}^2(\Omega)} \leq E_\varepsilon |\mathbb{P}(\Delta u)|^2_{\mathrm{L}^2(\Omega)}$$

for some boundary conditions (a mixture of Dirichlet, periodic and free). We believe that we can find a version of this inequality for our choice of boundary conditions and domain, however, due to time restrictions, this is left for now as an open-ended problem, with a view to future research.

4.3. **(III) term.** We now move our attention to the third term in (1.18),

$$\begin{aligned} \mu\langle\mathbb{P}(\Delta\mathbf{u}), \Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} &= \mu\langle\mathbb{P}(\Delta\mathbf{u}), \mathbb{P}(\Delta\mathbf{u})\rangle_{\mathrm{L}^2(\Omega)}, \\ &= \mu\|\mathbb{P}(\Delta\mathbf{u})\|^2_{\mathrm{L}^2(\Omega)} \end{aligned}$$

we can use the Pythagorean theorem to find

$$\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = \|\mathbb{P}(\Delta\mathbf{u})\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbb{Q}(\Delta\mathbf{u})\|^2_{\mathrm{L}^2(\Omega)}$$

$$\Rightarrow \qquad \mu\langle\mathbb{P}(\Delta\mathbf{u}), \Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = \mu\langle\Delta\mathbf{u}, \Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} - \mu\langle\nabla q_s, \nabla q_s\rangle_{\mathrm{L}^2(\Omega)} \; \leq \|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)},$$

which is due to $\mathbb{P}$ and $\mathbb{Q}$ being orthogonal, as seen in (1.13).

In [7] we see that, for an O(1) domain with Dirichlet boundary conditions, one can bound the Stokes pressure to the Laplacian of $\mathbf{u}$. By (1.15) in [7],

$$\int_\Omega |\nabla q_s|^2 \leq (\frac{1}{2} + \delta) \int_\Omega |\Delta\mathbf{u}|^2 + C \int_\Omega |\nabla\mathbf{u}|^2$$

and by (2.2),

$$\int_\Omega \nabla q_s \cdot \nabla\phi = \int_\Omega (\Delta\mathbf{u} - \nabla\nabla\cdot\mathbf{u}) \cdot \nabla\phi \forall\phi \in H^1(\Omega).$$

We look to establish a bound of the Stokes pressure, such that it is dominated by $\Delta\mathbf{u}$.

<u>**Without boundary conditions**</u>. We want to try and find some form of bound on $\|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)}$. We remember the properties (1.16) for q_s,

$$\begin{cases} \Delta q_s = 0, & \text{On } \Omega \\ \dfrac{\partial q_s}{\partial \vec{n}} = -\Delta \mathbf{u} \cdot \vec{n}, & \text{On } \Gamma \\ \displaystyle\int_\Omega q_s \, dx = 0, & \end{cases}$$

We also note that $\nabla \cdot (q_s \nabla q_s) = q_s(\Delta q_s) + \nabla q_s \cdot \nabla q_s$ which, when integrated over Ω, gives us

$$\begin{aligned} \langle \nabla q_s, \nabla q_s \rangle_{\mathrm{L}^2(\Omega)} &= -\int_\Omega q_s \Delta q_s + \int_\Omega \nabla \cdot (q_s \nabla q_s) \\ &= \int_\Gamma q_s (\vec{n} \cdot \nabla q_s) \\ &= -\int_\Gamma q_s (\Delta \mathbf{u} \cdot \vec{n}) \end{aligned}$$

$$\begin{aligned} \Delta \mathbf{u} &= -\nabla \times (\nabla \times \mathbf{u}) \\ \Rightarrow \quad (\Delta \mathbf{u} \cdot \vec{n}) &= -\nabla \times \omega \cdot \vec{n} \\ &= -\mathrm{div}(\omega \times \vec{n}) \end{aligned}$$

so,

$$\begin{aligned} \langle \nabla q_s, \nabla q_s \rangle_{\mathrm{L}^2(\Omega)} &= -\int_\Gamma q_s (\Delta \mathbf{u} \cdot \vec{n}) \\ &= \int_\Gamma q_s \, \mathrm{div}_\Omega (\omega \times \vec{n}). \end{aligned}$$

In this case; $\mathrm{div}_\Omega(\omega \times \vec{n})$ is the 3D divergence on the boundary, which depends on the value of $\omega \times \vec{n}$ on Ω near to the boundary, as opposed to looking at the value on the boundary. However,

Proposition 4.1. Let $\mathbf{v} \in \mathbf{C}^\infty$ be a smooth vector defined on Ω, and let $\vec{n}$ be the normal unit vector on Γ. Then, on the boundary,

$$\mathrm{div}_\Omega(\vec{n} \times \mathbf{v}) = \mathrm{div}_\Gamma(\vec{n} \times \mathbf{v}),$$

where div_Γ is the 2D divergence on the surface of Γ.

Proof. In spherical polar coordinates, the divergence of a vector $\mathbf{v}$ can be written as

$$\mathrm{div}_\Omega\,\mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2\mathbf{v}_r)}{\partial r} + \frac{1}{r\,\sin\theta}\frac{\partial(\sin\theta\,\mathbf{v}_\theta)}{\partial\theta} + \frac{1}{r\,\sin\theta}\frac{\partial(\mathbf{v}_\phi)}{\partial\phi},$$

where $\mathbf{v}_r$ represents the radial component of $\mathbf{v}$. Since $\vec{n}\times\mathbf{v}$ is always orthogonal to $\vec{n}$, $(\vec{n}\times\mathbf{v})_r = 0$ everywhere in Ω. Therefore,

$$\begin{aligned}\mathrm{div}_\Omega\,(\vec{n}\times\mathbf{v}) &= \frac{1}{r\,\sin\theta}\frac{\partial(\sin\theta\,(\vec{n}\times\mathbf{v})_\theta)}{\partial\theta} + \frac{1}{r\,\sin\theta}\frac{\partial((\vec{n}\times\mathbf{v})_\phi)}{\partial\phi}\\ &= \mathrm{div}_\Gamma\,(\vec{n}\times\mathbf{v})\end{aligned}$$

□

Hence, from Proposition 4.1 for Γ,

$$\langle\nabla q_s,\nabla q_s\rangle_{\mathrm{L}^2(\Omega)} = \int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n})$$

Thus, we have

$$\begin{aligned}\langle\mathbb{P}(\Delta\mathbf{u}),\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} &= \|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} - \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)},\\ \text{where}\quad \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)} &= \int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n})\end{aligned}\tag{4.6}$$

With 'Navier-Navier' boundary conditions. From (4.6), we have

$$\begin{aligned}\langle\mathbb{P}(\Delta\mathbf{u}),\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} &= \|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} - \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)},\\ \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)} &= \int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n}).\end{aligned}$$

Using our boundary conditions, we look to find a bound for $\|\nabla q_s\|$.

From Proposition (4.1) and the result found in (2.1b) for Γ,

$$\begin{aligned}\langle\nabla q_s,\nabla q_s\rangle_{\mathrm{L}^2(\Omega)} &= \int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n})\\ &= \int_\Gamma q_s\,\mathrm{div}_\Gamma(2\mathbf{u}\cdot\nabla\vec{n}).\end{aligned}$$

Then, by (2.2),

$$\begin{aligned}
\langle \nabla q_s, \nabla q_s \rangle_{\mathrm{L}^2(\Omega)} &= \int_{\Gamma_t} q_s \operatorname{div}_\Gamma(\frac{2}{r}\mathbf{u}) - \int_{\Gamma_b} q_s \operatorname{div}_\Gamma(\frac{2}{r}\mathbf{u}), \\
&= \int_\Gamma \int_{1-\varepsilon}^{1+\varepsilon} \frac{\partial}{\partial r}\Big(q_s \operatorname{div}_\Gamma(\frac{2}{r}\mathbf{u})\Big) \\
&= \int_\Omega \frac{\partial q_s}{\partial r} \operatorname{div}_\Gamma(\frac{2}{r}\mathbf{u}) + q_s \frac{\partial}{\partial r}\Big(\operatorname{div}_\Gamma(\frac{2}{r}\mathbf{u})\Big) \\
&\le C\Big[\|\nabla q_s\|_{\mathrm{L}^2(\Omega)} \|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} + \|q_s\|_{\mathrm{L}^2(\Omega)} \|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}\Big]
\end{aligned}$$

Thus, using the Poincaré inequality for q_s (8.4),

$$\begin{aligned}
\|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)} &\le C \|\nabla q_s\|_{\mathrm{L}^2(\Omega)} \Big[\|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} + \mathrm{O}(\varepsilon)\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}\Big] \\
&\le C^2 \Big[\|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} + \mathrm{O}(\varepsilon)\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}\Big]^2 \\
&\le C^2 \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} + \mathrm{O}(\varepsilon^2)\|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)}.
\end{aligned}$$

The last step is from using Young's inequality (8.3). We would now need to find a version of the Poincaré inequality for our 'Navier-Navier' boundary conditions in order to show that $\|\mathbf{u}\|_{\mathrm{H}^1(\Omega)} \le \|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}$. Due to a restriction on time, we move on to our final choice of boundary conditions.

With 'Non-slip - Navier' boundary conditions. As we saw for our previous scenarios,

$$\langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} = \|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} - \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)}.$$

For our choice of boundary conditions, there isn't a straightforward method for finding a bound of $\|q_s\|_{\mathrm{L}^2(\Omega)}$. However, in [7], for an O(1) smooth domain Ω and Dirichlet boundary conditions, it is shown that the Stokes pressure can be bounded by

$$\int_\Omega |\nabla q_s|^2 \le (\frac{1}{2} + \delta) \int_\Omega |\Delta \mathbf{u}|^2 + C \int_\Omega |\nabla \mathbf{u}|^2$$

where $C \ge 0$ is a constant, and $\delta > 0$ is a small parameter, such that $|\delta| << 1$. For our domain, we believe that we can bound the Stokes pressure to be $\mathrm{O}(\varepsilon)\|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}$ and, as such, can be absorbed into our Laplacian term. That is, for D a positive constant independent of ε

$$\begin{aligned}
-\langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u} \rangle_{\mathrm{L}^2(\Omega)} &= -\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)} \\
&\le -D\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}.
\end{aligned}$$

We wish to find our own version of Theorem 1.2 in [7], which is left for future research. For the rest of our thesis, we will assume that the inequality above is true. Hence,

$$-\mu\langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq -D\mu\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \tag{4.7}$$

5. Estimates for Leray Navier-Stokes equations

5.1. Without boundary conditions. We have our equation (1.18)

$$\langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathbb{P}(\mathbf{u}\cdot\nabla \mathbf{u}), \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} - \mu\langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = 0,$$

and then we saw that, without our boundary conditions

$$\begin{cases} \langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = -\dfrac{1}{2}\partial_t\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \displaystyle\int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot(\partial_t \mathbf{u}) & (4.1) \\ \langle \mathbb{P}(\mathbf{u}\cdot\nabla \mathbf{u}), \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq \dfrac{1}{2\mu}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)}\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \dfrac{\mu}{2}\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} & (4.4) \\ -\mu\langle \mathbb{P}(\Delta \mathbf{u}), \Delta \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = -\mu\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \mu\displaystyle\int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n}) & (4.6) \end{cases}$$

To find a bound for the norm of our gradient term, we return to Green's theorem, specifically

$$\begin{aligned} &\langle \nabla \mathbf{u}, \nabla \mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = \langle \Delta \mathbf{u}, -\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u}, \\ \Rightarrow\quad &\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq \|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u}, \qquad \text{(by Cauchy-Schwarz inequality (8.1))} \\ \Rightarrow\quad &\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq \frac{1}{2}\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u}, \qquad \text{(by Young's inequality (8.3))} \\ \Rightarrow\quad &-\frac{1}{2}\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u} \end{aligned}$$

Multiplying by μ, we then have

$$-\frac{\mu}{2}\|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\mu\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \mu\int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u} \tag{5.1}$$

So, collecting our results together, we find our ODE to be

$$\frac{1}{2}\partial_t\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\mu\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2\mu}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)}\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + I_\Gamma, \tag{5.2}$$

where

$$I_\Gamma = \int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot(\partial_t \mathbf{u}) + \mu\int_\Gamma (\vec{n}\cdot\nabla \mathbf{u})\cdot \mathbf{u} + \mu\int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n}).$$

5.2. **With 'Navier-Navier' boundary conditions.** With the introduction of our 'Navier-Navier' boundary conditions, we found

$$\begin{cases} \displaystyle\int_\Gamma (\vec{n}\cdot\nabla\mathbf{u})\cdot(\partial_t\mathbf{u}) = \pm\frac{1}{2r}\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} \\ \displaystyle\mu\int_\Gamma (\vec{n}\cdot\nabla\mathbf{u})\cdot\mathbf{u} = \pm\frac{\mu}{r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} \\ \displaystyle\mu\int_\Gamma q_s\,\mathrm{div}_\Gamma(\omega\times\vec{n}) \leq C^2\mu\|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} + \mathrm{O}(\varepsilon^2)\mu\|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)} \end{cases}$$

Now, our ODE becomes

$$\frac{1}{2}\partial_t\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2\mu}\|\mathbf{u}\|^2_{\mathrm{L}^\infty(\Omega)}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + I_\Gamma,$$

where

$$I_\Gamma \leq \pm\frac{1}{2r}\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} \pm \frac{\mu}{r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} + C^2\mu\|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} + \mathrm{O}(\varepsilon^2)\mu\|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)}.$$

5.3. **With 'Non-slip - Navier' boundary conditions.** The inclusion of the 'Non-slip - Navier' boundary conditions provided us with

$$\begin{cases} \langle\partial_t\mathbf{u},\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} = -\frac{1}{2}\partial_t\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2r}\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma)} & (4.3) \\ \langle\mathbb{P}(\mathbf{u}\cdot\nabla\mathbf{u}),\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} & (4.5) \\ -\mu\langle\mathbb{P}(\Delta\mathbf{u}),\Delta\mathbf{u}\rangle_{\mathrm{L}^2(\Omega)} \leq -\mu D\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} & (4.7) \\ -\frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -\mu\alpha\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} & (5.1) \end{cases}$$

The last result is from (5.1),

$$\begin{aligned} -\frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} &\leq -\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \mu\int_\Gamma(\vec{n}\cdot\nabla\mathbf{u})\cdot\mathbf{u} \\ &\leq -\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)} \\ &\leq -\mu\alpha\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \end{aligned}$$

where the final line is a result found in Subsection 3.2. We can also use the Poincaré inequality (8.4) to bound our $\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$ term, i.e.

$$\begin{aligned} -\frac{\mu}{2}\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} &\leq -\mu\alpha\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mu}{2}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \\ &\leq -\mu\alpha\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \mathrm{O}(\varepsilon^2)\frac{\mu}{2}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \\ &\leq -\mu\alpha\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}, \end{aligned}$$

Hence, the new form for our ODE is

$$\frac{1}{2}\partial_t\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq -E\alpha\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2r}\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)} + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)}. \tag{5.3}$$

where $E := D - \frac{1}{2}$.

6. **ODE analysis**

Integrating with respect to $t \in [0, T]$

$$\int_0^{\mathrm{T}} \frac{1}{2}\partial_t\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = \frac{1}{2}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(T) - \frac{1}{2}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(0)$$

$$\int_0^{\mathrm{T}} \frac{1}{2r}\partial_t\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)} = \frac{1}{2r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)}(T) - \frac{1}{2r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)}(0)$$

Putting these together, we find

$$\frac{1}{2}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}\Big|_0^{\mathrm{T}} \leq \frac{1}{2r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)}\Big|_0^{\mathrm{T}} + \int_0^{\mathrm{T}}\Big(-E\alpha\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)}\Big)$$

Thus,

$$\frac{1}{2}\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}\Big|_0^{\mathrm{T}} \leq -E\int_0^{\mathrm{T}} \alpha\mu\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\int_0^{\mathrm{T}}\|\nabla\mathbf{u}\|^{10}_{\mathrm{L}^2(\Omega)} + F.$$

where $F = \frac{1}{2r}\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)}\Big|_0^{\mathrm{T}}$.
Then, defining a variable $q \in \mathbb{R}^+$ such that,

$$\frac{1}{2}q\Big|_0^{\mathrm{T}} = -E\alpha\mu\int_0^{\mathrm{T}} q + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\int_0^{\mathrm{T}} q^5 + F$$

$$\frac{1}{2}\partial_t q = -E\alpha\mu q + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}q^5$$

By Grönwall's inequality, we then have

$$\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq q$$

So our ODE becomes,

$$\frac{1}{2}\frac{dq}{dt} = -E\alpha\mu q + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}\,q^5 =: f(q). \tag{6.1}$$

A stationary solution for f is when $\frac{df}{dq}(q) = 0$, so

$$\frac{df}{dq} = -E\alpha\mu + \frac{5\mathrm{O}(\varepsilon^2)}{\mu^7}\,q^4 = 0$$

$$\Rightarrow \qquad q^* = \Big(\frac{E\alpha\mu^8}{5\mathrm{O}(\varepsilon^2)}\Big)^{\frac{1}{4}} = \mathrm{O}\Big(\frac{\mu^2}{\varepsilon^{\frac{1}{2}}}\Big)$$

There are 3 other solutions for q^*, that being $-q^*$, iq^*, and $-iq^*$, however, for $q^* \in \mathbb{R}^+$, the solution above is the only appropriate solution. Hence, we know that, for $q \in \mathbb{R}^+$, there is only one stationary point. Also, since $f(0) > 0$ and $f(q) \to \infty$ as $q \to \infty$, we see that this stationary point must be a minimum point.

Hence, the minimum gradient of q is

$$\frac{1}{2}\frac{dq}{dt}_{min} = f\Big(\Big(\frac{E\alpha\mu^8}{5\mathrm{O}(\varepsilon^2)}\Big)^{\frac{1}{4}}\Big) = -E\alpha\mu q^* + \frac{E\alpha\mu}{5}q^*$$
$$= -\frac{4E\alpha\mu}{5}q^*$$

We want the gradient to be negative for some values of q, therefore we need

$$-\frac{4E\alpha\mu}{5}q^* < 0$$

Since $q \in \mathbb{R}^+$, the minimum gradient is always negative. Clearly, we can see that $q^- = 0$ is a solution to our ode. The other solutions are for q^+, where

$$-E\alpha\mu + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}(q^+)^4 = 0$$
$$q^+ = \Big(\frac{E\alpha\mu^8}{\mathrm{O}(\varepsilon^2)}\Big)^{\frac{1}{4}} = \mathrm{O}\Big(\frac{\mu^2}{\varepsilon^{\frac{1}{2}}}\Big)$$

Again, since $q^+ \in \mathbb{R}^+$, there is only one nonzero solution for our ode that is real and positive. For an arbitrary variation δq and solution $\tilde{q}$, such that $f(\tilde{q}) = 0$, we see that

$$f(\tilde{q}+\delta q) = \big[-E\alpha\mu\tilde{q} + \frac{\mathrm{O}(\varepsilon^2)}{\mu^7}(\tilde{q})^5 + \mathrm{O}(\varepsilon^2)\mu p\big] - E\alpha\mu\delta q + \frac{5\mathrm{O}(\varepsilon^2)}{\mu^7}(\tilde{q})^4\delta q + \mathrm{O}(\delta q^2)$$
$$\approx \big(-E\alpha\mu + \frac{5\mathrm{O}(\varepsilon^2)}{\mu^7}(\tilde{q})^4\big)\delta q + \mathrm{O}(\delta q^2)$$

For δq a small variation, $\delta q > (\delta q)^2$, therefore

$$f(\tilde{q}+\delta q) \propto \big(-E\alpha\mu + \frac{5\mathrm{O}(\varepsilon^2)}{\mu^7}(\tilde{q})^4\big)\delta q$$

This means that, for $q^- = 0$,

$$(q^-)^4 < \frac{E\alpha\mu^8}{5\mathrm{O}(\varepsilon^2)}$$
$$f(\delta q) \propto \big(-E\alpha\mu\big)\delta q = -\kappa^2\delta q$$

for some $\kappa \in \mathbb{R}$.

Similarly, for $q^+ = \left(\frac{E\alpha\mu^8}{\mathrm{O}(\varepsilon^2)}\right)^{\frac{1}{4}}$,

$$(q^+)^4 > \frac{E\alpha\mu^8}{5\mathrm{O}(\varepsilon^2)}$$

$$f(q^+ + \delta q) \propto \big(- E\alpha\mu + +5E\alpha\mu\big)\delta q = \upsilon^2 \delta q$$

for some $\upsilon \in \mathbb{R}$.

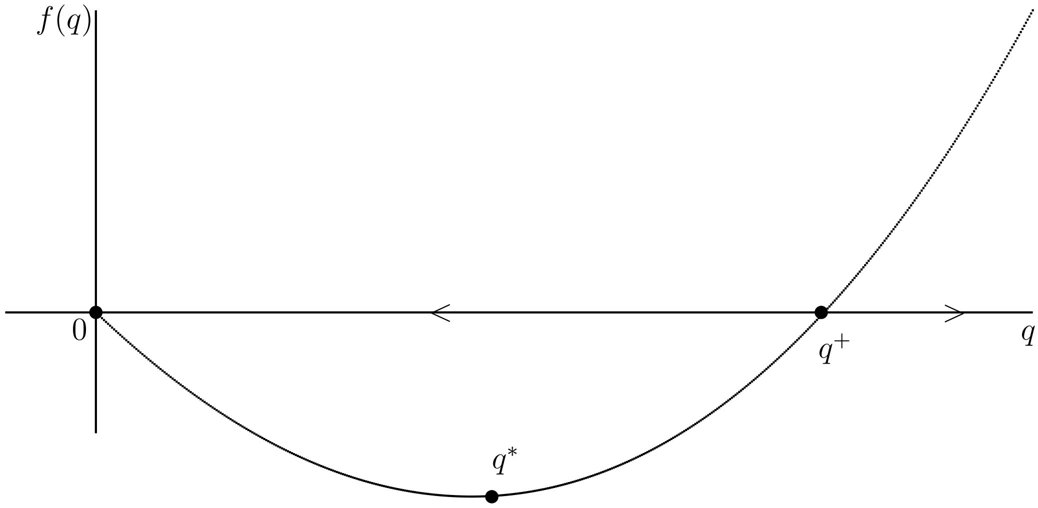

FIGURE 2. Phase portrait of $\frac{dq}{dt} = f(q)$, with respect to q

Therefore, we have that values of q near our q^- steady state are attracted towards q^-, while those near q^+ are repelled away. Hence, q^- is a stable steady state and q^+ is unstable, as shown in Figure 2.

This means that, as long as we choose initial data $\|\nabla \mathbf{u}_0\|_{\mathrm{L}^2(\Omega)} < \|\nabla \mathbf{u}^+\|_{\mathrm{L}^2(\Omega)}$, then $\|\nabla \mathbf{u}\|_{\mathrm{L}^2(\Omega)}(t) < \|\nabla \mathbf{u}^+\|_{\mathrm{L}^2(\Omega)}$ $\forall t \in \mathbb{R}^+$,

7. Conclusion

Throughout this project, we have worked with our Navier boundary conditions (generalised, 'Navier-Navier', and 'Non-slip - Navier') on a thin spherical shell in order to find restrictions for the existence of a solution to the incompressible Navier-Stokes equations (1.3).

Following our work on the matter, we have found that there exist positive constants $c_1, c_2, c_3, \varepsilon_0$, independent of ε, such that, for any $0 < \varepsilon \leq \varepsilon_0$, with initial data

$$\mathbf{u}_0 \in \mathrm{V} := \{\mathbf{v} \in \mathrm{L}^2(\Omega) | \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \vec{n}\big|_{\Gamma} = 0, \mathbf{v}|_{\Gamma_b} = 0, [\boldsymbol{S}\vec{n}]_{\tan}|_{\Gamma_t} = \vec{0}\},$$

where

$$\|\nabla \mathbf{u}_0\|^2_{\mathrm{L}^2(\Omega)} \leq c_1 \varepsilon^{-\frac{1}{2}}.$$

Then, the incompressible Navier-Stokes equations (1.3) have a solution $\mathbf{u} \in \mathrm{V}$, such that

$$\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}(t) \leq c_2 \|\mathbf{u}_0\|_{\mathrm{L}^2(\Omega)},$$
$$\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}(t) \leq c_3 \varepsilon^{-\frac{1}{2}}, \qquad \text{for all}\, t \geq 0.$$

Hence we have an existence theorem for the incompressible Navier-Stokes equations on thin spherical shell, with our 'Non-slip - Navier' boundary conditions.

7.1. **Future work.** For future research, we would look to follow the example of R. Temam and M. Ziane's paper [3], utilising their 'N**u**, M**u**' decomposition (where M**u** is the average of the flow and N**u** the complement of M**u**) for our 'Navier-Navier' boundary conditions. This includes exploring their treatment of the case when $\mathbf{u} = \vec{A} + \vec{B} \times \mathbf{x}$, as well as further investigating the proof of Agmon's inequality (8.7), aiming to prove $\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)}$ is bounded by $\|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}$ for our domain and boundary conditions. We also want to establish a bound for the Stokes pressure, similar to the one seen in [7], such that it is dominated by the Laplacian of **u**.

Bounding the H2 norm by the Laplacian operator. Finding a bound

$$\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)} \leq \mathrm{O}(\varepsilon^p) \|\Delta \mathbf{u}\|_{\mathrm{L}^2(\Omega)}$$

for $p \geq 0$, is integral to finding accurate estimates for our domain and particular choice of boundary conditions. As a framework for future research, we consult different pieces of literature in order to derive a method for our problem. In [2], it is found that, for a thin (non-spherical) domain with Navier boundary conditions,

$$\|\nabla^2 \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} = \int_\Omega |\Delta \mathbf{u}|^2 + \sum_{i,j=1}^{3} \int_\Gamma \partial_j \cdot (N_i \partial_j - N_j \partial_i) \partial_i \mathbf{u}.$$

Then, after finding estimates for the boundary terms,

$$\|\nabla^2 \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq \|\Delta \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + C_0 \varepsilon \|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)} + C \|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)} \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)}.$$

Thus, we then have a bound for $\|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)}$. Another approach is seen in [6], where we are working on a thin spherical domain with boundary conditions different from our own choices. In this case, for a function $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$

$$\sum_{i,j=1}^{2} \int_{\Omega} \Big|\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\Big|^2 = \int_{\Omega} |\Delta\varphi|^2 + \int_{\Gamma} \operatorname{tr}\mathfrak{B}[\gamma(\nabla\varphi)\cdot\vec{n}]^2.$$

Here, $\mathfrak{B}$ is the second fundamental form, $\mathfrak{B}(\zeta_1, \zeta_2) = -(\zeta_1 \cdot \nabla)\vec{n} \cdot \zeta_2$, where ζ_1, ζ_2 are tangent vectors. γ is the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma)$, and $\operatorname{tr}\mathfrak{B}$ represents the matrix trace of the second fundamental form. However, to find an estimate between $\|\nabla^2\mathbf{u}\|$ and $\|\Delta\mathbf{u}\|$, the paper assumes the domain is convex, while we feel that although our domain near to Γ_t is locally convex, we cannot assume our domain is convex as we approach Γ_b. Hence, although we believe that [6] provides us with an insightful method for dealing with this problem, we feel there are extra steps needed in order to conclude that $\|\mathbf{u}\|^2_{\mathrm{H}^2(\Omega)} \leq C\|\Delta\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)}$. These papers do, however, provide inspiration for future research.

8. Appendix

8.1. **Useful inequalities.** We will use these inequalities throughout the paper.

Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v}\rangle_{\mathrm{L}^2(\Omega)}| \leq \|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\|\mathbf{v}\|_{\mathrm{L}^2(\Omega)} \tag{8.1}$$

Hölder's inequality:

$$\|\mathbf{u}\cdot\mathbf{v}\|_{\mathrm{L}^1(\Omega)} \leq \|\mathbf{u}\|_{\mathrm{L}^p(\Omega)}\|\mathbf{v}\|_{\mathrm{L}^q(\Omega)}, \tag{8.2}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{8.3}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Poincaré inequality:

Let $\mathbf{u} \in \{\mathbf{u} \in \mathrm{L}^2(\Omega) \cap \mathrm{H}^1(\Omega) |\, \mathrm{div}(\mathbf{u}) = 0\,,\, \mathbf{u}\cdot\vec{n}\big|_{\Gamma} = 0,\, \mathbf{u}|_{\Gamma_b} = \mathbf{0}\}$ and $0 \leq \varepsilon \leq \frac{1}{2}$. Then,

$$\|\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \leq \mathrm{O}(\varepsilon^2)\|\nabla\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} \tag{8.4}$$

8.2. **Proof of the Poincaré inequality for the Stokes pressure on our thin domain.** From our definition of q_s, we know that our Stokes pressure has zero spatial average on Ω,

$$\int_\Omega q_s = 0.$$

Defining the spatial average of q_s on S^2 as

$$\bar{q}_s := \frac{1}{\mathrm{Area}(S^2)}\int_{S^2} q_s \, sin\,\theta \;\mathrm{d}\theta\,\mathrm{d}\phi,$$

we now look to find a version of Poincaré inequality in Ω for q_s.

$$\begin{aligned}
\|q_s\|^2_{\mathrm{L}^2(\Omega)} &= \int\int\int_\Omega r^2\, sin\,\theta\; q_s^2\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{1-\varepsilon}^{1+\varepsilon} r^2\Big(\int_{S^2} q_s^2\Big)\,\mathrm{d}r \\
&\leq \int_{1-\varepsilon}^{1+\varepsilon} r^2\Big(\int_{S^2}\big(q_s - \frac{1}{|S^2|}\int_{S^2} q_s\big)^2 + \big(\frac{1}{|S^2|}\int_{S^2} q_s\big)^2\Big)\,\mathrm{d}r \\
&\leq \mathrm{O}(1)\int_{1-\varepsilon}^{1+\varepsilon} r^2\bar{q}_s^2\,\mathrm{d}r.
\end{aligned}$$

For a general function $p = p(t)$, where p has zero spatial average,

$$\int_0^1 p^2\,\mathrm{d}t \leq \mathrm{O}(1)\int_0^1 \Big|\frac{\partial p}{\partial t}\Big|^2\,\mathrm{d}t.$$

Then, defining $n = 2\varepsilon t$,

$$\begin{aligned}
&\int_0^{2\varepsilon} p^2\,\frac{\mathrm{d}n}{2\varepsilon} \leq \mathrm{O}(1)\int_0^{2\varepsilon}\Big|2\varepsilon\frac{\partial p}{\partial n}\Big|^2\,\frac{\mathrm{d}n}{2\varepsilon}, \\
\Rightarrow \quad &\int_0^{2\varepsilon} p^2\,\mathrm{d}n \leq \mathrm{O}(\varepsilon^2)\int_0^{2\varepsilon}\Big|\frac{\partial p}{\partial n}\Big|^2\,\mathrm{d}n.
\end{aligned}$$

Now, returning to our integral and defining $r' := r - (1-\varepsilon)$

$$
\begin{aligned}
\int_{1-\varepsilon}^{1+\varepsilon} r^2 \bar{q}_s^2 \,\mathrm{d}r &= \int_0^{2\varepsilon} (1-\varepsilon+r')^2\, \bar{q}_s^2 \,\mathrm{d}r' \\
&\le \int_0^{2\varepsilon} (1-\varepsilon)^2\, \bar{q}_s^2 \,\mathrm{d}r' \;+ \int_0^{2\varepsilon} (r')^2\, \bar{q}_s^2 \,\mathrm{d}r' \\
&\le \mathrm{O}(1) \int_0^{2\varepsilon} \bar{q}_s^2 \,\mathrm{d}r' \;+ \int_0^{2\varepsilon} (r'\, \bar{q}_s)^2 \,\mathrm{d}r' \\
&\le \mathrm{O}(\varepsilon^2) \int_0^{2\varepsilon} \Big|\frac{\partial \bar{q}_s}{\partial r'}\Big|^2 \,\mathrm{d}r' \;+ \mathrm{O}(\varepsilon^2) \int_0^{2\varepsilon} \Big|\frac{\partial (r'\, \bar{q}_s)}{\partial r'}\Big|^2 \,\mathrm{d}r' \\
&\le \mathrm{O}(\varepsilon^2) \int_0^{2\varepsilon} \big(1+(r')^2\big)\Big|\frac{\partial \bar{q}_s}{\partial r'}\Big|^2 \,\mathrm{d}r' \;+ \mathrm{O}(\varepsilon^2) \int_0^{2\varepsilon} \bar{q}_s{}^2 \,\mathrm{d}r' \\
&\le \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} \Big|\frac{\partial \bar{q}_s}{\partial r}\Big|^2 \,\mathrm{d}r \;+ \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} \bar{q}_s{}^2 \,\mathrm{d}r
\end{aligned}
$$

We can also write

$$
\begin{aligned}
\mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} \bar{q}_s{}^2 \,\mathrm{d}r &\le \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} (1-r)^2\, \bar{q}_s{}^2 \,\mathrm{d}r + \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} r^2\, \bar{q}_s{}^2 \,\mathrm{d}r \\
&\le \mathrm{O}(\varepsilon^5) \int_{1-\varepsilon}^{1+\varepsilon} \bar{q}_s{}^2 \,\mathrm{d}r + \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} r^2\, \bar{q}_s{}^2 \,\mathrm{d}r
\end{aligned}
$$

Then, by bounding the second term to our original integral, we can conclude

$$
\begin{aligned}
\big(1-\mathrm{O}(\varepsilon^2)\big) \int_{1-\varepsilon}^{1+\varepsilon} r^2 \bar{q}_s^2 \,\mathrm{d}r &\le \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} \Big|\frac{\partial \bar{q}_s}{\partial r}\Big|^2 \,\mathrm{d}r \;+ \mathrm{O}(\varepsilon^5) \int_{1-\varepsilon}^{1+\varepsilon} \bar{q}_s{}^2 \,\mathrm{d}r \\
\Rightarrow \qquad \int_{1-\varepsilon}^{1+\varepsilon} r^2 \bar{q}_s^2 \,\mathrm{d}r &\le \mathrm{O}(\varepsilon^2) \int_{1-\varepsilon}^{1+\varepsilon} \Big|\frac{\partial \bar{q}_s}{\partial r}\Big|^2 \,\mathrm{d}r \le \mathrm{O}(\varepsilon^2) \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)}
\end{aligned}
$$

And thus, we have found

$$
\|q_s\|^2_{\mathrm{L}^2(\Omega)} \le \mathrm{O}(\varepsilon^2) \|\nabla q_s\|^2_{\mathrm{L}^2(\Omega)} \tag{8.5}
$$

8.3. **Bounding $\|\nabla \mathbf{u}\|$ by $\|\Delta \mathbf{u}\|$.** In order to work towards proving Agmon's inequality (8.7), we first need to find

$$
\Big\|\frac{\partial \mathbf{u}}{\partial r}\Big\|_{\mathrm{L}^2(\Omega)} \le \mathrm{O}(\varepsilon) \Big\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\Big\|_{\mathrm{L}^2(\Omega)}. \tag{8.6}
$$

To show this, we first use Green's theorem,

$$
\begin{aligned}
\int_\Omega \frac{\partial \mathbf{u}}{\partial r} \cdot \frac{\partial \mathbf{u}}{\partial r} &= -\int_\Omega \frac{\partial^2 \mathbf{u}}{\partial r^2} \cdot \mathbf{u} + \int_\Gamma (\vec{n} \cdot \frac{\partial \mathbf{u}}{\partial r}) \cdot \mathbf{u}, \\
&\le \Big\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\Big\|_{\mathrm{L}^2(\Omega)} \|\mathbf{u}\|_{\mathrm{L}^2(\Omega)} + \frac{1}{r} \|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)}.
\end{aligned}
$$

As we saw in the derivation of (3.1),

$$\|\mathbf{u}\|^2_{\mathrm{L}^2(\Gamma_t)} \leq \mathrm{O}(\varepsilon)\|\frac{\partial \mathbf{u}}{\partial r}\|^2_{\mathrm{L}^2(\Omega)}.$$

Therefore, we can rearrange to find

$$\begin{aligned}\|\frac{\partial \mathbf{u}}{\partial r}\|^2_{\mathrm{L}^2(\Omega)} &\leq D\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\|_{\mathrm{L}^2(\Omega)}\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)} \\ &\leq \mathrm{O}(\varepsilon)\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\|_{\mathrm{L}^2(\Omega)}\|\frac{\partial \mathbf{u}}{\partial r}\|_{\mathrm{L}^2(\Omega)}, \qquad \text{from Poincaré inequality (8.4).}\end{aligned}$$

Hence, dividing by $\|\frac{\partial \mathbf{u}}{\partial r}\|_{\mathrm{L}^2(\Omega)}$, we obtain (8.6)

$$\|\frac{\partial \mathbf{u}}{\partial r}\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{O}(\varepsilon)\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\|_{\mathrm{L}^2(\Omega)}.$$

8.4. **Agmon's inequality for our thin domain.** We will use a result from within Lemma 2.2 in [3] as our version of Agmon's inequality. Using Temam & Ziane's version of Agmon's inequality established in [6],

$$\begin{aligned}\|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \leq c_0\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\prod_{i=1}^{2}\Big(\|\frac{\partial^2 \mathbf{u}}{\partial x_i^2}\|_{\mathrm{L}^2(\Omega)} + \|\frac{\partial \mathbf{u}}{\partial x_i}\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\Big)^{\frac{1}{4}} \\ \times\Big(\|\frac{\partial^2 \mathbf{u}}{\partial x_3^2}\|_{\mathrm{L}^2(\Omega)} + \frac{1}{2\varepsilon}\|\frac{\partial \mathbf{u}}{\partial x_3}\|_{\mathrm{L}^2(\Omega)} + \frac{1}{4\varepsilon^2}\|\mathbf{u}\|_{\mathrm{L}^2(\Omega)}\Big)^{\frac{1}{4}}.\end{aligned}$$

Then, using our version of the Poincaré inequality (8.4), and our result (8.6)

$$\|\frac{\partial \mathbf{u}}{\partial r}\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{O}(\varepsilon)\|\frac{\partial^2 \mathbf{u}}{\partial r^2}\|_{\mathrm{L}^2(\Omega)},$$

we find that, for our flow $\mathbf{u}$

$$\|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \leq c_0\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\|\mathbf{u}\|^{\frac{3}{4}}_{\mathrm{H}^2(\Omega)}$$

We would like to be able to show that

$$\|\mathbf{u}\|_{\mathrm{H}^2(\Omega)} \leq \|\Delta\mathbf{u}\|_{\mathrm{L}^2(\Omega)},$$

so that we can then write

$$\|\mathbf{u}\|_{\mathrm{L}^\infty(\Omega)} \leq c_0\|\mathbf{u}\|^{\frac{1}{4}}_{\mathrm{L}^2(\Omega)}\|\Delta\mathbf{u}\|^{\frac{3}{4}}_{\mathrm{L}^2(\Omega)} \tag{8.7}$$

where c_0 is independent of ε. Due to time restrictions, we look to [2] and [6] as potential framework for how to solve this problem.

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